

Decomposition of some Reshetikhin-Turaev representations into irreducible factors

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Abstract

We give the decomposition into irreducible factors of the $SU(2)$ Reshetikhin-Turaev representations of the mapping class group of surfaces when the level is $p = 4r$ or $p = 2r^2$ with r an odd prime or when $p = 2r_1r_2$ with r_1, r_2 two distinct odd primes, under certain technical assumptions.

Keywords: Reshetikhin-Turaev representations, mapping class group, quantum representations, Topological Quantum Field Theory.

1 Introduction

Witten gave in [Wit89] convincing arguments for the existence of Topological Field Theories, as defined in [Ati88, Wit88], giving a three dimensional interpretation of the Jones polynomial when the gauge group is $SU(2)$. Each of these TQFTs gives a family of projective finite dimensional representations of the mapping class group $\text{Mod}(\Sigma_g)$ of a genus g closed oriented surface Σ_g . Reshetikhin and Turaev gave a rigorous construction of these TQFTs [RT91] using representations of quantum groups. In this paper we will follow the skein theoretical construction of [Lic91, BHMV95] to define these representations.

We can lift these projective representations to linear representations of some central extension $\widetilde{\text{Mod}}(\Sigma_g)$ of $\text{Mod}(\Sigma_g)$ noted:

$$\rho_{p,g} : \widetilde{\text{Mod}}(\Sigma_g) \rightarrow \text{GL}(V_{p,g}).$$

Here $p = 2(k+2) \geq 3$ is an even integer indexing the representations and $V_{p,g}$ is a finite dimensional complex vector space. These representations are equipped with an invariant scalar product $\langle \cdot, \cdot \rangle_{p,g}$ with respect to which they are unitary.

The goal of this paper is to decompose some of these representations into irreducible factors. Only few results are known concerning their decomposition. In [BHMV95], an explicit proper submodule of $V_{p,g}$ is given whenever 4 divides p . In [Rob01] it is shown that $V_{p,g}$ is irreducible when $\frac{p}{2}$ is an odd prime. Robert's proof extends word-by-word to show that the modules $V_{18,g}$ are also irreducible. In [AF10] the authors showed that for $p = 24, 36, 60$ then $V_{p,g}$ contains at least three invariant submodules. Finally we gave in [Kor13] an explicit decomposition into irreducible factors of the modules $V_{p,1}$ for arbitrary level $p \geq 3$.

The main results of this paper are summarized in the two following theorems:

Theorem 1.1.

1. If r is an odd prime, then $V_{4r,2}$ is the sum of two irreducible subrepresentations.
2. If r is an odd prime, then $V_{2r^2,2}$ is irreducible.
3. If r_1, r_2 are two distinct odd primes, then $V_{2r_1r_2,2}$ is irreducible.

Given a level $p = 2r \geq 3$, there exists a set of complex numbers called $6j$ -symbols at level p which will be defined in the next section. If r is odd, we call *generic* a level for which none of the level p $6j$ -symbols is null. When r is even, we exhibit in Proposition 3.10 two families of vanishing $6j$ -symbols at level p . We call such a p *generic* if no other $6j$ -symbols vanish. Numerical computations suggest that every levels are generic.

Theorem 1.2.

1. The modules $V_{18,g}$ are irreducible for arbitrary $g \geq 2$.
2. If 50 is generic then the module $V_{50,3}$ is irreducible.
3. If r is an odd prime, $p = 4r$ is generic and $g = 3$, then $V_{4r,3}$ is sum of two irreducible subrepresentations.
4. If r_1, r_2 are two distinct odd primes, $p = 2r_1r_2$ is generic and $2g < \min(r_1, r_2)$, then $V_{2r_1r_2,g}$ is irreducible.

The genericity of a given level p can be checked by numerical computations. This leads us to the following:

Corollary 1.3. (*Computer assisted proof*)

The module $V_{50,3}$ is irreducible. The modules $V_{28,g}$ for $3 \leq g \leq 4$, $V_{44,g}$ for $3 \leq g \leq 8$ and $V_{58,g}$ for $3 \leq g \leq 10$ are sum of two irreducible submodules.

Remark. In [BHMV95] some representations $\rho_{p,g}$ are also defined when p is odd. They verify $\rho_{2p,g} \cong \rho_{p,g} \otimes \rho_{6,g}$. In particular if an odd level r is such that $V_{2r,g}$ is irreducible, then so is $V_{r,g}$. This extends the two previous theorems to the $SO(3)$ cases as well.

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2 Skein construction of the Reshetikhin-Turaev representations

Following [BHMV95], we will briefly define the representations $\rho_{p,g}$ and fix some notations.

2.1 The spaces $V_{p,g}$

Given an even integer $p \geq 6$, we denote by $A \in \mathbb{C}$ an arbitrary primitive $2p$ -th roots of unity. Using the Kauffman skein relation of Figure 1, we associate to any framed link $L \subset S^3$ an invariant $\langle L \rangle_p \in \mathbb{C}$.

$$\begin{array}{c} \text{Crossing} = A \left(\text{Vertical lines} \right) + A^{-1} \left(\text{Horizontal lines} \right) \\ \text{Thick circle} = -(A^2 + A^{-2}) \text{Thick circle with diagonal line} \end{array}$$

Figure 1: Skein relations defining the framed link invariants.

Choose $g \geq 1$ and denote by C_g the set of isotopy classes of framed links (including the empty link) in an oriented genus g handlebody H_g . We fix a genus g Heegaard splitting of the sphere, i.e.

an element $S \in \text{Mod}(\Sigma_g)$ and two handlebodies so that :

$$H_g^1 \bigcup_{S: \partial H_g^1 \rightarrow \partial H_g^2} H_g^2 \cong S^3$$

Take $L_1, L_2 \in C_g$ and embed L_1 in H_g^1 and L_2 in H_g^2 . The above gluing defines a link $L_1 \bigcup_S L_2 \subset S^3$. We call Hopf pairing the bilinear form:

$$(\cdot, \cdot)_{g,p}^H : \mathbb{C}[C_g] \times \mathbb{C}[C_g] \rightarrow \mathbb{C}$$

defined by

$$(L_1, L_2)_{p,g}^H := \left\langle L_1 \bigcup_S L_2 \right\rangle_p$$

Eventually we define the spaces $V_{p,g}$ as the quotients:

$$V_{p,g} := \mathbb{C}[C_g] / \ker \left((\cdot, \cdot)_{g,p}^H \right)$$

The vector spaces $V_{p,g}$ are finite dimensional ([BHMV95]) and we can find explicit basis as follows. Let $g \geq 2$, choose a trivalent graph $\Gamma \subset H_g$ so that H_g retracts on Γ by deformation. If $g = 1$, Γ represents the circle $S^1 \times \{0\} \subset S^1 \times D^2 \cong H_1$. We denote by $E(\Gamma)$ the set of its edges.

A triple $(i, j, k) \in \{0, \dots, \frac{p-4}{2}\}^3$ is said *p-admissible* if:

1. $|i - j| \leq k \leq i + j$,
2. $i + j + k$ is even and is smaller or equal to $p - 4$.

A map $\sigma : E(\Gamma) \rightarrow \{0, \dots, \frac{p-4}{2}\}$ is a *p-admissible coloring* of Γ if for every three edges $e_1, e_2, e_3 \in E(\Gamma)$ adjacent to a vertex, the triple $(\sigma(e_1), \sigma(e_2), \sigma(e_3))$ is *p-admissible*.

In [Jon83, Wen87] the authors defined some idempotents $\{f_0, \dots, f_{\frac{p-4}{2}}\}$ of the Temperley-Lieb algebra with coefficient in $\mathbb{Q}(A)$ called Jones-Wenzl idempotents. To σ a *p-admissible coloring* of Γ we associate a vector $u_\sigma \in V_{p,g}$ as follows. We replace each edge $e \in E(\Gamma)$ by the Jones-Wenzl idempotent $f_{\sigma(e)}$. If (e_1, e_2, e_3) are three edges adjacent to a vertex of Γ , we connect the idempotents using the link $T_{\sigma(e_1), \sigma(e_2), \sigma(e_3)}$ defined in Figure 2.

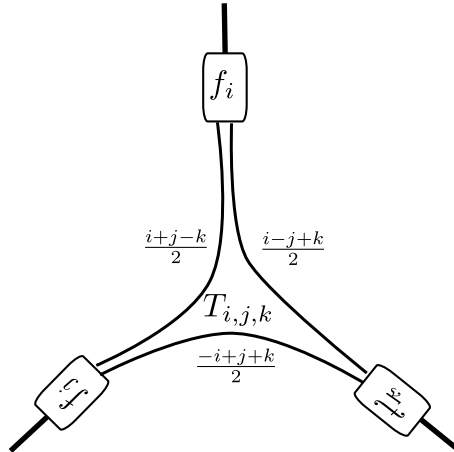


Figure 2: The link $T_{i,j,k}$ used to connect three idempotents f_i, f_j and f_k . The numbers above each three arcs denotes the number of parallel copies of the arc used to define the link.

Theorem ?? asserts that the elements u_σ , for σ a *p-admissible coloring* of Γ , form a basis of $V_{p,g}$. Moreover there exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{p,g}$ on $V_{p,g}$ invariant under the action of $\text{Mod}(\widetilde{\Sigma_g})$, for which the vectors u_σ are pairwise orthogonal.

The basis u_σ depends on the choice of the embedded trivalent graph. We can transform a trivalent graph into one another by a sequence of Whitehead moves. Suppose that Γ_1 and Γ_2 are two trivalent graphs of genus $g \geq 2$, which only differ by a single Whitehead move, inside a ball B^3 , as drawn in Figure 3.

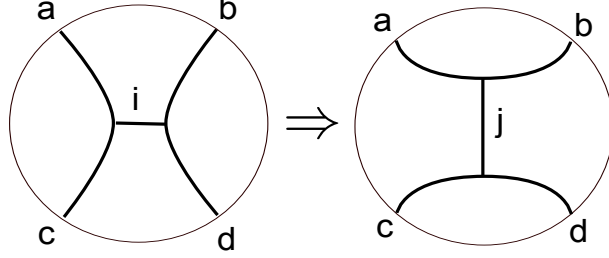


Figure 3: The two graphs Γ_1 on the left and Γ_2 on the right differ by a local Whitehead move.

Fix a p -admissible coloring of the graphs outside B^3 and denote by $\begin{array}{c} \diagup \sigma(i) \diagdown \\ \diagdown \end{array}$ (resp. $\begin{array}{c} \diagup \sigma(j) \diagdown \\ \diagdown \end{array}$) the vector associated to the coloration of Γ_1 (resp of Γ_2) with the edge i colored by $\sigma(i)$ (resp with the edge j colored by $\sigma(j)$).

Then the vectors $\begin{array}{c} \diagup i \diagdown \\ \diagdown \end{array}$ belong to the subspace spanned by the vectors $\begin{array}{c} \diagup j \diagdown \\ \diagdown \end{array}$ and decompose using the so-called 'fusion rules' formula

Lemma 2.1.

$$\begin{array}{c} \diagup i \diagdown \\ \diagdown \end{array} = \sum_j \left\{ \begin{array}{ccc} a & b & j \\ c & d & i \end{array} \right\} \begin{array}{c} \diagup j \diagdown \\ \diagdown \end{array}$$

where the sum runs through p -admissible colorings and the coefficient $\left\{ \begin{array}{ccc} a & b & j \\ c & d & i \end{array} \right\}$ only depends on the colors of the edges a, b, c, d, i and j and is called *recoupling coefficient* or *6j-symbol* in literature. We refer to [MV94] for a proof and an explicit computation of these coefficients.

2.2 The Reshetikhin-Turaev representations

We fix an orientation preserving homeomorphism

$$\alpha : \Sigma_g \rightarrow \partial H_g$$

Choose a class $\phi \in \text{Mod}(\Sigma_g)$ associated to a homeomorphism which extends to H_g through α . Then ϕ acts on C_g and preserves the kernel of the Hopf pairing so acts on $V_{p,g}$ by passing to the quotient. Denote by $\tilde{\rho}_{p,g}(\phi) \in \text{GL}(V_{p,g})$ the resulting operator.

Now choose $\phi \in \text{Mod}(\Sigma_g)$ so that the corresponding homeomorphisms extend to H_g through $\alpha \circ S$. This extension also defines, by quotient, an operator on $V_{p,g}$. We denote by $\tilde{\rho}_{p,g}(\phi)$ the dual of this operator for the Hopf pairing.

The elements of $\text{Mod}(\Sigma_g)$ which extend to H_g either through α or through $\alpha \circ S$, generate the whole group $\text{Mod}(\Sigma_g)$. It is a non trivial fact that the associated operators $\tilde{\rho}_{p,g}(\phi)$ generate a projective representation:

$$\tilde{\rho}_{p,g} : \text{Mod}(\Sigma_g) \rightarrow \text{PGL}(V_{p,g})$$

We consider a central extension $\widetilde{\text{Mod}(\Sigma_g)}$ of $\text{Mod}(\Sigma_g)$ that lifts the above projective representations to linear ones (see [MR95, GM13]):

$$\rho_{p,g} : \widetilde{\text{Mod}(\Sigma_g)} \rightarrow \text{GL}(V_{p,g})$$

These are the so-called Reshetikhin-Turaev representations.

Now to each edge $e \in E(\Gamma)$, choose a disc D_e , properly embedded in H_g , that intersects Γ transversely once in e . Note that the set of boundary curves $\gamma_e := \partial D_e \subset \partial H_g \xrightarrow{\alpha} \Sigma_g$ forms a pants decomposition of Σ_g .

A classical property of the Jones-Wenzl idempotents (Lemma ??) asserts that, if $T_e \in \text{Mod}(\Sigma_g)$ denotes the Dehn twist along γ_e , then:

$$\tilde{\rho}_{p,g}(T_e) \cdot u_\sigma = \mu_{\sigma(e)} u_\sigma$$

where $\mu_i := (-1)^i A^{i(i+2)}$.

We fix the lift of T_e in $\widetilde{\text{Mod}}(\Sigma_g)$, still denoted T_e , so that $\rho_{p,g}(T_e) \cdot u_\sigma = \mu_{\sigma(e)} u_\sigma$.

We also fix the lift $S \in \widetilde{\text{Mod}}(\Sigma_g)$ so that the matrix of $\rho_{p,g}(S)$ is the matrix of the Hopf pairing $(\cdot, \cdot)_{p,g}^H$ multiplied by an element $\eta \in \mathbb{C}$ which verifies $|\eta| = \frac{|A^2 - A^{-2}|}{\sqrt{p}}$. We refer to [BHMV95], where η represents the quantum invariant of S^3 , for a detailed discussion on η .

Since S and the $\{T_e\}_{e \in E(\Gamma)}$ generate $\widetilde{\text{Mod}}(\Sigma_g)$ for some trivalent graphs, we have an explicit description of $\rho_{p,g}$.

3 Cyclicity of the vacuum vector

Denote by $\mathcal{A}_{p,g}$ the subalgebra of $\text{End}(V_{p,g})$ generated by the operators $\rho_{p,g}(\phi)$ for $\phi \in \widetilde{\text{Mod}}(\Sigma_g)$. The key ingredient to prove Theorem 1.1 is to show that the vacuum vector $v_0 \in V_{p,g}$, associated to the class of the empty link, is cyclic, i.e. that $\mathcal{A}_{p,g} \cdot v_0 = V_{p,g}$.

3.1 The genus one case

In [Kor13] we gave an explicit decomposition of the Weil representations into irreducible factors. An easy generalization of the arguments of the proof of Lemma 3 of [FK06] leads to an explicit isomorphism of $SL_2(\mathbb{Z})$ -modules between $V_{p,1}$ and the odd submodule of the Weil representation at level p . Proving that $v_0 \in V_{p,1}$ is cyclic reduces to show that its projection on each irreducible submodule of $V_{p,1}$ is not null.

Denote by $\{u_0, \dots, u_{\frac{p-4}{2}}\}$ the basis of $V_{p,1}$ where u_i is the class of the closure of the i -th Jones-Wenzl idempotent along a longitude in H_1 . Also denote by $\{e_i, i \in \mathbb{Z}/p\mathbb{Z}\}$ the basis of the Weil $SL_2(\mathbb{Z})$ -module U_p at level p as described in [Kor13].

In this basis, the Weil projective representations in genus one are defined by the matrices:

$$\begin{aligned} \pi_p(S) &= \frac{1}{\sqrt{p}} (A^{-ij})_{i,j \in \mathbb{Z}/p\mathbb{Z}} \\ \pi_p(T) &= (A^{i^2} \delta_{i,j})_{i,j \in \mathbb{Z}/p\mathbb{Z}} \end{aligned}$$

Here the level is an integer $p \geq 2$ not necessary even. When p is even, we take A to be a primitive $2p$ -th roots of unity. When p is odd, A is a primitive p -th roots of unity.

The vectors $\{e_i^- := e_i - e_{-i}, i \in \{1, \dots, \frac{p-2}{2}\}\}$ span a submodule $U_p^- \subset U_p$.

Lemma 3.1. *Let $p = 2r \geq 6$ be an even integer. Then the following map:*

$$\Psi : \begin{cases} U_p^- & \rightarrow V_{p,1} \\ e_i^- & \mapsto u_{i+r-1} \end{cases}$$

is an isomorphism of $SL_2(\mathbb{Z})$ -projective modules.

Proof. We compute the matrix elements:

$$\begin{aligned}\langle \psi(e_j^-), \rho_{p,1}(S)\psi(e_i^-) \rangle &= \frac{\eta \cdot (-1)^{i+j}}{A^2 - A^{-2}} \left(A^{2(i+r)(j+r)} - A^{-2(i+r)(j+r)} \right) \\ &= \frac{\eta \cdot (-1)^r \sqrt{p}}{A^2 - A^{-2}} \langle e_j^-, \pi_p(S)e_i^- \rangle\end{aligned}$$

where the scalar $\frac{\eta \cdot (-1)^r \sqrt{p}}{A^2 - A^{-2}}$ has norm one.

$$\begin{aligned}\langle \psi(e_j^-), \rho_{p,1}(T)\psi(e_i^-) \rangle &= (-1)^{i+r-1} A^{(i+r-1)(i+r+1)} \delta_{i,j} \\ &= (-1)^{r-1} A^{-1} \langle e_j^-, \pi_p(T)e_i^- \rangle\end{aligned}$$

□

The decomposition into irreducible submodules of U_p is described by the following:

Proposition 3.2 ([Kor13]). *We have the following decompositions where \cong denotes an isomorphism of $SL_2(\mathbb{Z})$ -modules:*

1. If a and b are coprime, then $U_{ab} \cong U_a \otimes U_b$.
2. If r is prime and $n \geq 1$, then $U_{r^{n+2}} \cong U_{r^n} \oplus W_{r^{n+2}}$ where $W_{r^{n+2}}$ denotes another module.
3. If r is an odd prime, then $U_{r^2} \cong \mathbb{1} \oplus W_{r^2}$ where $\mathbb{1}$ is the trivial representation.
4. The modules U_p for $r > 2$ and W_{r^n} split into two submodules: $U_p \cong U_p^- \oplus U_p^+$, $W_{r^n} \cong W_{r^n}^+ \oplus W_{r^n}^-$.
5. The modules $B_1 \otimes \dots \otimes B_k$, where the B_i have the form $U_r^+, U_r^-, U_2, U_4^+, U_4^-, W_{r^n}^+$ or $W_{r^n}^-$ and have pairwise coprime levels, are irreducible.

We can now prove:

Proposition 3.3. *Let $p \geq 6$ be an even integer. Then the vacuum vector $v_0 \in V_{p,1}$ is cyclic if and only if one of the following three cases holds:*

- $p = 2r_1 \dots r_k$ with r_i distinct odd primes.
- $p = 2r^2$ with r prime.
- $p = 4r$ with r prime.

Proof. We will use Proposition 3.2 and the explicit isomorphisms given in the main theorem of [Kor13] to study whether the vector

$$v := \psi^{-1}(v_0) = e_{\frac{p-2}{2}} - e_{\frac{p+2}{2}} \in U_p^-$$

has non trivial projection on each submodule of U_p^- or not.

Given two integers x and n , we will denote by $[x]_n \in \mathbb{Z}/n\mathbb{Z}$ the class of x modulo n . We write

$$v = e_{[x]_p} - e_{[-x]_p} \in U_p^-$$

with $x = \frac{p-2}{2}$.

First, when $p = 2r^2$, with r prime, the module U_p^- is irreducible so the vector is cyclic.

When $p = 4r$, with r an odd prime, the module decomposes into two irreducible submodules:

$$U_{4r}^- \cong U_4^- \otimes U_r^+ \oplus U_4^+ \otimes U_r^-$$

The vector v decomposes as follows:

$$\begin{aligned}v &= e_{[x]_4} \otimes e_{[x]_r} + e_{[-x]_4} \otimes e_{[-x]_r} \\ &= \left(\frac{1}{2}(e_{[x]_4} - e_{[-x]_4}) \otimes (e_{[x]_r} + e_{[-x]_r}) \right) \\ &\quad + \left(\frac{1}{2}(e_{[x]_4} + e_{[-x]_4}) \otimes (e_{[x]_r} - e_{[-x]_r}) \right)\end{aligned}$$

Where the first term lies in $U_4^- \otimes U_r^+$ and the second in $U_4^+ \otimes U_r^-$.

Since $x = 2r - 1$, neither 4 nor r divide x , so $[x]_4 \neq [x]_4$ and $[x]_r \neq [x]_r$ and the two projections are not null.

When $p = 2r_1 \dots r_k$, with r_i distinct odd primes, we have the following decomposition:

$$U_p^- \cong \bigoplus_{\epsilon = (\epsilon_i)_{i \in \{-1, +1\}^k}} X_\epsilon$$

where

$$X_\epsilon := U_2 \otimes U_{r_1}^{\epsilon_1} \otimes \dots \otimes U_{r_k}^{\epsilon_k}$$

Let us fix ϵ and denote:

$$e_\epsilon := e_{[x]_2} \otimes e_{[x]_{r_1}}^{\epsilon_1} \otimes \dots \otimes e_{[x]_{r_k}}^{\epsilon_k} \in X_\epsilon$$

where we used the notation $e_i^\pm := e_i \pm e_{-i}$. By using the facts that $\langle e_i, e_i^\epsilon \rangle = 1$ and $\langle e_{-i}, e_i^\epsilon \rangle = (-1)^{\frac{1-\epsilon}{2}}$, we compute:

$$\begin{aligned} \langle v, e_\epsilon \rangle &= \left\langle e_{[x]_2} \otimes e_{[x]_{r_1}} \otimes \dots \otimes e_{[x]_{r_k}}, e_{[x]_{r_1}}^{\epsilon_1} \otimes \dots \otimes e_{[x]_{r_k}}^{\epsilon_k} \right\rangle \\ &\quad - \left\langle e_{[x]_2} \otimes e_{[-x]_{r_1}} \otimes \dots \otimes e_{[-x]_{r_k}}, e_{[x]_{r_1}}^{\epsilon_1} \otimes \dots \otimes e_{[x]_{r_k}}^{\epsilon_k} \right\rangle \\ &= 1 - (-1)^{\sum_i \frac{1-\epsilon_i}{2}} = 2 \neq 0 \end{aligned}$$

So the projection of v on each irreducible submodule X_ϵ is not null.

Now suppose that $p = 2r_1^{n_1} \dots r_k^{n_k}$ with $k \geq 2$, r_i distinct primes and $n_1 \geq 2$. Since r_1 does not divide x , the vector v has a null projection on the submodule:

$$\begin{cases} U_{r_1^{n_1-2}}^+ \otimes U_{2r_2^{n_2} \dots r_k^{n_k}}^-, & \text{if } r_1 \neq 2, \\ U_{2^{n_1-1}}^+ \otimes U_{r_2^{n_2} \dots r_k^{n_k}}^-, & \text{if } r_1 = 2 \end{cases}$$

Next if $p = 2r^n$, with r an odd prime and $n \geq 2$, the projection of v on $U_2 \otimes U_{r^{n-2}}^-$ is null.

Finally if $p = 2^n$, with $n \geq 3$, the projection of v on $U_{2^{n-2}}^-$ is null. \square

3.2 Cyclicity in higher genus

We will denote by $Z_{p,g}$ the subspace of $V_{p,g}$ defined by:

$$Z_{p,g} := \text{Span}\{u_\sigma, \text{ so that } (u_\sigma, v_0)^H \neq 0\}$$

The goal of this subsection is to prove the following:

Proposition 3.4. *When $g \geq 2$, we have:*

1. *When $p = 4r$ with r an odd prime and if $g = 2$ or if p is generic and $g < r - 2$.*
2. *When $p = 2r^2$ with r an odd prime and $g = 2$ or if $p = 50$ is generic and $g = 3$, then the vacuum vector $v_0 \in V_{p,g}$ is cyclic.*
3. *When $p = 2r_1 r_2$ with r_1, r_2 distinct odd primes and $g = 2$ or if p is generic and $2g < \min(r_1, r_2)$, then the vacuum vector $v_0 \in V_{p,g}$ is cyclic.*
4. *When $p = 4r$ with r an odd prime and if $g = 2$ or if p is generic, and $g < r - 2$, then $Z_{p,g}$ is included in the cyclic subspace generated by v_0 .*

Fix a trivalent graph $\Gamma \subset H_g$ as in section 2. Two p -admissible colorings σ_1, σ_2 of Γ will be said equivalent if:

$$(-1)^{\sigma_1(e)} A^{\sigma_1(e)(\sigma_1(e)+2)} = (-1)^{\sigma_2(e)} A^{\sigma_2(e)(\sigma_2(e)+2)}, \text{ for all } e \in E(\Gamma)$$

We denote by $\underline{col}_p(\Gamma)$ the set of equivalence classes of colorings for this relation. To $[\sigma] \in \underline{col}_p(\Gamma)$, we associate the subspace:

$$W_{[\sigma]} := \text{Span}\{u_{\sigma'}, \sigma' \in [\sigma]\} \subset V_{p,g}$$

Lemma 3.5. *If $X \subset V_{p,g}$ is a $\widetilde{\text{Mod}(\Sigma_g)}$ -submodule, then:*

$$X = \bigoplus_{[\sigma] \in \underline{col}_p(\Gamma)} X \cap W_{[\sigma]}$$

Proof. The matrices $\rho_{p,g}(T_e)$, for $e \in E(\Gamma)$, generate a commutative subalgebra of $\mathcal{A}_{p,g}$. The set $\underline{col}_p(\Gamma)$ indexes its characters and the spaces $W_{[\sigma]}$ are the associated common eigenspaces of the $\rho_{p,g}(T_e)$. The orthogonal projector on X must commute with the $\rho_{p,g}(T_e)$ and thus preserves the subspaces $W_{[\sigma]}$. \square

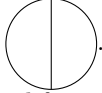
The strategy to prove Proposition 3.4 is to apply Lemma 3.5 to

$$X := (\mathcal{A}_{p,g} \cdot v_0)^\perp$$

the orthogonal (for the invariant form) of the cyclic space generated by the vacuum vector.

Definition 3.6.

1. We call Γ_g a *fly eyes graph* of genus g if it is a trivalent graph obtained by the following inductive method:

- Γ_2 is the Theta graph .
- A graph Γ_{g+1} is obtained from a Γ_g by choosing arbitrary a vertex and inserting a triangle as drawn on the left-hand side of Figure 4.

The right-hand side gives an example of a genus 8 fly eyes graph.

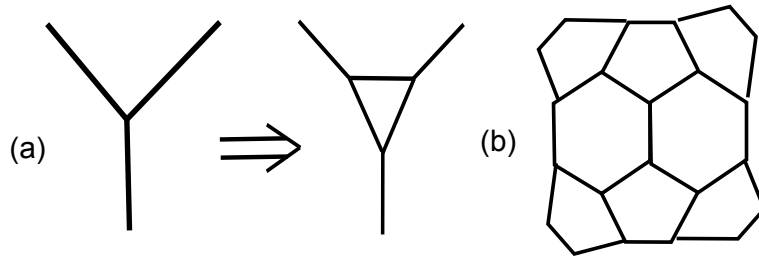


Figure 4: On the left: the operation transforming a fly eyes graph of genus g into a one of genus $g + 1$. On the right: an example of genus 8 fly eye graph.

2. The genus 3 fly eyes graph is unique and is called the tetrahedron graph. We say that a level of the form $p = 2r$, with r odd, is generic if for any coloring σ of Γ_3 , we have:

$$(u_\sigma, v_0)_{p,3}^H \neq 0$$

The complex numbers $(u_\sigma, v_0)_{p,3}^H$ are called tetrahedron coefficients in literature and are related to the $6j$ -symbols defined in the previous section. In particular it is equivalent to say that the $6j$ -symbols or the tetrahedron coefficients are not null for a level p . It follows from fusion-rules (equation (2.1)) that if p is generic, then for any $g \geq 3$, for any fly eyes graph Γ_g and for any p -admissible coloring σ of Γ_g , we have:

$$(u_\sigma, v_0)_{p,3}^H \neq 0$$

Fix $g \geq 2$ and embed a fly eyes graph Γ_g in S^3 . Denote by H_g the embedded handlebody

$$H_g := S^3 \setminus V(\Gamma_g)$$

where $V(\Gamma_g)$ denotes a tubular neighborhood of Γ_g . For each edge $e \in E(\Gamma_g)$, fix a curve $\gamma_e \subset H_g$ which bounds a disc intersecting Γ_g only once along e .

We construct a map:

$$w : \mathbb{N}^{E(\Gamma_g)} \rightarrow V_{p,g}$$

as follows. To $f : E(\Gamma_g) \rightarrow \mathbb{N}$ we associate the class in $V_{p,g}$ of the link made of $f(e)$ parallel copies of γ_e for each edge $e \in E(\Gamma_g)$.

When $g = 2$, we will note $w_{a,b,c} \in V_{p,2}$ the class of the link made of a parallel copies of γ_1 , b copies of γ_2 and c of γ_3 .

Figure 5 shows the curves γ_e when $g = 2$ and $g = 3$.

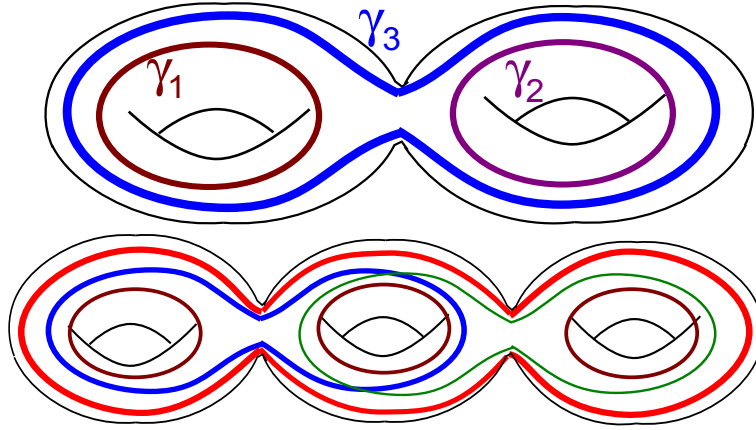


Figure 5: The curves γ_e defining the map w are drawn when $g = 2$ and $g = 3$.

Lemma 3.7. *If $p = 4r$, with r an odd prime, or if $p = 2r_1r_2$, with r_1, r_2 two distinct odd primes, then:*

$$w_{a,b,c} \in \mathcal{A}_{p,2} \cdot v_0, \text{ for any } a, b, c \in \{0, 1\}$$

Proof. Choose a longitude L and a meridian M of Σ_1 . The space $\mathcal{A}_{p,1} \cdot v_0 = V_{p,1}$ is generated by juxtaposition of properly embedded parallel copies of L and M in H_1 , colored by the element $\omega \in V_{p,1}$ as defined in [BHMV92].

By embedding the skein elements $L(\omega)$ and $M(\omega)$ in $H_2 \cong H_1 \# H_1$ in both handles, we see that the vectors $w_{i,0,j}$ belong to $\mathcal{A}_{p,2} \cdot v_0$ for arbitrary i and j . So do the vectors $w_{i,j,0}$ by action of the mapping class group.

It remains to show that $w_{1,1,1} \in \mathcal{A}_{p,2} \cdot v_0$. It follows from the definition of Jones-Wenzl idempotents that:

$$w_{1,1,1} = 2 \left(\begin{array}{c|c} & \\ \hline 2 & \\ \hline & \end{array} \right) 2 + w_{2,0,0} + w_{0,2,0} + (A^2 + A^{-2})v_0$$

Thus we just have to show that $2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} 2 \in \mathcal{A}_{p,2} \cdot v_0$.

Using Lemma 2.1, we have that:

$$\begin{aligned} 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} 2 &= \sum_{k=0,2,4} \left\{ \begin{matrix} 2 & 2 & k \\ 2 & 2 & 0 \end{matrix} \right\} \begin{matrix} 2 & & 2 \\ \circ & \text{---} k & \circ \end{matrix} \\ \rho_{p,2}(T_e) \cdot 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} 2 &= \sum_{k=0,2,4} \left\{ \begin{matrix} 2 & 2 & k \\ 2 & 2 & 0 \end{matrix} \right\} \mu_k \begin{matrix} 2 & & 2 \\ \circ & \text{---} k & \circ \end{matrix} \end{aligned}$$

where T_e is (a lift of) the Dehn twist around the middle edge of the Theta graph (labeled 0).

Since both vectors belong to $V_{p,2} \cdot v_0$ and since the recoupling coefficients $\left\{ \begin{matrix} 2 & 2 & k \\ 2 & 2 & 0 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 2 & 2 & k \\ 2 & 2 & 2 \end{matrix} \right\}$ are not zero and $\mu_2 \neq 1$, we know that the 3-dimensional space generated by $\begin{matrix} 2 & & 2 \\ \circ & \text{---} 0 & \circ \end{matrix}$, $\begin{matrix} 2 & & 2 \\ \circ & \text{---} 2 & \circ \end{matrix}$ and $\begin{matrix} 2 & & 2 \\ \circ & \text{---} 4 & \circ \end{matrix}$ is included in $\mathcal{A}_{p,2} \cdot v_0$. So does the vector

$$2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} 2. \quad \square$$

Lemma 3.8. *When $p = 2r^2$, with r an odd prime, then*

$$w_{a,b,c} \in \mathcal{A}_{p,2} \cdot v_0, \text{ for all } 0 \leq a, b, c \leq \frac{r-3}{2}.$$

Moreover, if σ is a p -admissible coloring of $\Gamma = \begin{pmatrix} \circ & \text{---} \circ \end{pmatrix}$ such that:

$$\sigma(e) \not\equiv -1 \pmod{r}, \text{ for all } e \in E(\Gamma)$$

then $u_\sigma \in \mathcal{A}_{p,2} \cdot v_0$.

Proof. Note first that $i, j \in \{0, \dots, \frac{p-4}{2}\}$ are such that:

- $\mu_i = \mu_j$,
- $i \neq j$,

if and only if $i \equiv j \equiv -1 \pmod{r}$ and i and j have same parity (and are distinct). Thus when σ satisfies the condition of the lemma, the subspace $W_{[\sigma]}$ is one-dimensional. Lemma 3.5 implies that this subspace is either in $\mathcal{A}_{p,2} \cdot v_0$, or in its orthogonal. Now note that the Hopf pairing $(u_\sigma, v_0)_{p,2}^H$ is not zero for it is equal to a $3j$ -symbol. This prove the second part of the lemma.

In particular, we just proved that:

$$\text{Span} \left(\begin{matrix} u & & w \\ \circ & \text{---} v & \circ \end{matrix}, 0 \leq u, v, w \leq r-2 \right) \subset \mathcal{A}_{p,2} \cdot v_0$$

We finish the proof by noticing that the vector $w_{a,b,c}$ belongs to this space whenever we have:

$$\begin{cases} a + c \leq r-2 \\ b + c \leq r-2 \\ a + b \leq r-2 \end{cases}$$

\square

Lemma 3.9. *The vector w_f belongs to $\mathcal{A}_{p,g} \cdot v_0$ for $f \in \{0, 1\}^{E(\Gamma_g)}$ when p is generic and:*

- $p = 4r$ with r an odd prime such that $g \leq r - 2$.
- $p = 2r_1r_2$ with r_1, r_2 distinct odd primes and $2g \leq \min(r_1, r_2)$.
- $p = 50$ and $g = 3$.

Proof. We proceed like in the proof of Lemma 3.8: first we note that if $f \in \{0, 1\}^{E(\Gamma_g)}$ then:

$$w_f \in \text{Span}(u_\sigma, 0 \leq \sigma(e) \leq g \text{ for all } e \in E(\Gamma_g))$$

Then we note that if σ is such that $0 \leq \sigma(e) \leq g$ for all $e \in \Gamma_g$, then $W_{[\sigma]}$ is one-dimensional so is included in $\mathcal{A}_{p,g} \cdot v_0$ for $(u_\sigma, v_0)_{p,g}^H \neq 0$ by assumption. The fact that these $W_{[\sigma]}$ are one-dimensional is deduced from the following two facts:

1. When $p = 4r$, and $i, j \in \{0, \dots, \frac{p-4}{2}\}$, then $\mu_i = \mu_j$ if and only if:
 - $i = j$,
 - or $i = \frac{p-4}{2} - j$ and i is even.
2. When $p = 2r_1r_2$, and $i, j \in \{0, \dots, \frac{p-4}{2}\}$, then $\mu_i = \mu_j$ if and only if:
 - $i = j$,
 - or j is the only element satisfying
$$\begin{cases} i \equiv j & (\text{mod } 2r_1) \\ i \equiv -j - 2 & (\text{mod } r_2) \end{cases} \quad \text{or} \quad \begin{cases} i \equiv j & (\text{mod } 2r_2) \\ i \equiv -j - 2 & (\text{mod } r_1) \end{cases}$$

□

Proof of Proposition 3.4. Fix a fly eyes graph Γ , a class $[\sigma] \in \underline{\text{col}}_p(\Gamma)$, and choose a vector

$$v = \sum_{\sigma' \in [\sigma]} \alpha_{\sigma'} u_{\sigma'} \in W_{[\sigma]} \cap (\mathcal{A}_{p,g} \cdot v_0)^\perp$$

By Lemma 3.5, we must show that $v = 0$ to conclude. We will find $\dim(W_{[\sigma]})$ independent equations verified by the coefficients $\alpha_{\sigma'}$.

Note $F \subset \mathbb{N}^{E(\Gamma)}$ the set of functions f so that:

- $f(e) \in \{0, 1\}, \forall e \in E(\Gamma)$, if $p = 4r$ or $p = 2r_1r_2$,
- $f(e) \in \{0, \dots, \frac{r-3}{2}\}, \forall e \in E(\Gamma)$, if $p = 2r^2$.

Using Lemmas 3.7, 3.8 and 3.9, we know that

$$w_f \in \mathcal{A}_{p,g} \cdot v_0, \text{ for all } f \in F$$

By definition of v , we have that:

$$(w_f, v_0)_{p,g}^H = 0, \quad \text{for all } f \in F \quad (1)$$

$$\Leftrightarrow \sum_{\sigma' \in [\sigma]} \left(\prod_{e \in E(\Gamma)} \lambda_{\sigma'(e)}^{f(e)} \right) \alpha_{\sigma'} (u_{\sigma'}, v_0)_{p,g}^H = 0, \quad \text{for all } f \in F \quad (2)$$

where $\lambda_i = -(A^{2(i+1)} + A^{-2(i+1)})$.

Since the complex numbers $(u_{\sigma'}, v_0)_{p,g}^H$ are non null when $p = 2r^2$ or $p = 2r_1r_2$ is generic or when $p = 4r$ and $u_\sigma \in Z_{p,g}$, it is enough to show that the matrix:

$$M := \left(\prod_{e \in E(\Gamma)} \lambda_{\sigma'(e)}^{f(e)} \right)_{\substack{\sigma' \in [\sigma] \\ f \in F}}$$

has independent lines to conclude the proof.

We now define an invertible square matrix \tilde{M} such that M is obtained from \tilde{M} by removing some lines.

When $i \in \{0, \dots, \frac{p-4}{2}\}$ we define the set:

$$\omega(i) := \left\{ j \in \left\{ 0, \dots, \frac{p-4}{2} \right\}, \text{ so that } \mu_i = \mu_j \right\}$$

And the Vandermonde matrix:

$$N[i] := (\lambda_j^n)_{\substack{j \in \omega(i) \\ 0 \leq n \leq \#\omega(i)-1}}$$

Since $\lambda_i \neq \lambda_j$ when $i \neq j \in \omega(i)$, the matrix $N[i]$ is invertible.

Now note $E(\Gamma) = \{e_1, \dots, e_{3g-3}\}$ and choose $\sigma \in [\sigma]$ arbitrary. The matrix

$$\tilde{M} := N(e_1) \otimes \dots \otimes N(e_{3g-3})$$

is clearly invertible and M is obtained from \tilde{M} by removing the lines corresponding to non p -admissible colorings of Γ . \square

3.3 Null $6j$ -symbols when 4 divides p

When $p = 2r$ with r odd, numerical computations suggest that there is no null $6j$ -symbols at level p . On the contrary, when 4 divides p , we have two families of $6j$ -symbols that vanish at level p :

Proposition 3.10. *Suppose 4 divides $p \geq 8$ and write $p = 2(k+2)$, with k an even integer. Then the following tetrahedron coefficients vanish:*

1. *Type I:*

$$\left\langle \begin{array}{c} \frac{k}{2} \\ a \quad b \quad c \end{array} \right\rangle = 0$$

when $a + b + c \equiv 2 \pmod{4}$.

2. *Type II:*

$$\left\langle \begin{array}{c} \frac{k}{2} \\ a \quad k-a \\ b \quad c \quad b \end{array} \right\rangle = 0$$

when $a + \frac{c+k}{2} \equiv 1 \pmod{2}$.

Definition 3.11. If 4 divides $p \geq 8$, we say that p is generic if the only vanishing $6j$ -symbols at level p are the ones given in Proposition 3.10

Numerical computations shows us that every level $p \leq 50$ is generic. We conjecture that every level is generic.

Lemma 3.12. *Let a, b be two integers such that $(a, k-a, b)$ is p -admissible. Set*

$$F(a, b) := \frac{\left\langle \begin{array}{c} a \\ k \quad k-a \\ b \quad a \end{array} \right\rangle}{\left\langle \begin{array}{c} k-a \quad a \\ b \end{array} \right\rangle}$$

Then we have $F(a, b) = (-1)^{\frac{b+k}{2}+a}$.

Proof. A straightforward computation using the formulas of [MV94] gives:

$$F(a, b) = \frac{f(a)}{g(b)}, \quad \text{where} \quad f(a) = (-1)^a [a+1]! [k-a]!$$

$$g(b) = (-1)^{\frac{k+b}{2}} \left[\frac{k-b}{2} \right]! \left[\frac{k+b}{2} + 1 \right]!$$

We now remark that

$$\frac{f(a+1)}{f(a)} = -\frac{[a+2]}{[k-a]} = -1$$

and

$$\frac{g(b+2)}{g(b)} = -\frac{[\frac{k+b}{2}+2]}{[\frac{k-b}{2}]} = -1$$

We conclude using the fact that $F(\frac{k}{2}, 2) = -1$. □

Lemma 3.13. *If (a, b, c) is a p -admissible triple, then we have:*

$$\langle k-a \rangle \langle k-b \rangle \frac{\left\langle \begin{array}{c} a \\ k \quad b \\ c \quad k-b \end{array} \right\rangle \left\langle \begin{array}{c} k-a \\ k \quad k-b \\ a \quad c \end{array} \right\rangle}{\left\langle \begin{array}{c} k-a \quad k-b \\ c \end{array} \right\rangle \left\langle \begin{array}{c} a \quad k-b \\ c \end{array} \right\rangle} = 1$$

Proof. We use the fact that adding a trivial ribbon colored by k does not change the class of a vector. We work in the space associated to the sphere with three punctures colored by a, b and c :

$$\begin{aligned}
& \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ | \\ c \end{array} = \begin{array}{c} a \quad k \quad b \\ \diagdown \quad \diagup \\ \bigcirc \\ | \\ c \end{array} \\
& = \langle k-a \rangle \langle k-b \rangle \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ k \\ \diagdown \quad \diagup \\ k-a \quad k-b \\ \diagdown \quad \diagup \\ a \quad b \\ | \\ c \end{array} \\
& = \langle k-a \rangle \langle k-b \rangle \frac{\left\langle \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ k \\ \diagdown \quad \diagup \\ k-a \quad k-b \\ \diagdown \quad \diagup \\ k-a \quad k-b \\ | \\ c \end{array} \right\rangle \left\langle \begin{array}{c} k-a \quad b \\ \diagdown \quad \diagup \\ k \\ \diagdown \quad \diagup \\ a \quad k-b \\ \diagdown \quad \diagup \\ a \quad k-b \\ | \\ c \end{array} \right\rangle}{\left\langle \begin{array}{c} k-a \quad k-b \\ \diagdown \quad \diagup \\ c \end{array} \right\rangle \left\langle \begin{array}{c} a \quad k-b \\ \diagdown \quad \diagup \\ c \end{array} \right\rangle} \cdot \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ | \\ c \end{array}
\end{aligned}$$

We conclude by identifying both vectors. \square

Proof of Proposition 3.10. We use the fact that the Kauffman bracket of a link in S^3 does not change if we add a trivial ribbon colored by k . First when σ is of type *I*, we use Lemma 2.1 and Lemma 3.12 to compute:

$$\begin{aligned}
& \begin{array}{c} \frac{k}{2} \\ \bigcirc \\ \diagdown \quad \diagup \\ \frac{k}{2} \quad \frac{k}{2} \\ a \quad b \quad c \end{array} = \langle \frac{k}{2} \rangle^3 \begin{array}{c} \frac{k}{2} \\ \bigcirc \\ \diagdown \quad \diagup \\ k \quad k \\ \frac{k}{2} \quad \frac{k}{2} \\ a \quad b \quad c \end{array} \\
& = F\left(\frac{k}{2}, a\right) F\left(\frac{k}{2}, b\right) F\left(\frac{k}{2}, c\right) \begin{array}{c} \frac{k}{2} \\ \bigcirc \\ \diagdown \quad \diagup \\ \frac{k}{2} \quad \frac{k}{2} \\ a \quad b \quad c \end{array} \\
& = - \begin{array}{c} \frac{k}{2} \\ \bigcirc \\ \diagdown \quad \diagup \\ \frac{k}{2} \quad \frac{k}{2} \\ a \quad b \quad c \end{array} \\
& \text{Thus } \left\langle \begin{array}{c} \frac{k}{2} \\ \bigcirc \\ \diagdown \quad \diagup \\ \frac{k}{2} \quad \frac{k}{2} \\ a \quad b \quad c \end{array} \right\rangle = 0.
\end{aligned}$$

When σ is of type *II*, a similar computation using Lemmas 3.12 and 3.13 gives:

$$\begin{aligned}
& \left(\text{Diagram 1} \right) = \langle \frac{k}{2} \rangle \langle a \rangle \langle k-a \rangle \\
& = F(a, c) \cdot \left(\langle \frac{k}{2} \rangle \langle k-a \rangle \frac{ \left(\text{Diagram 2} \right) \cdot \left(\text{Diagram 3} \right) }{ \left(\text{Diagram 4} \right) \cdot \left(\text{Diagram 5} \right) } \right) \left(\text{Diagram 6} \right) \\
& = - \left(\text{Diagram 7} \right) \\
& \text{Thus } \left(\text{Diagram 8} \right) = 0. \quad \square
\end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 2:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 3:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 4:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 5:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 6:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 7:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.
- Diagram 8:** A circle with a central vertex connected to three points on the boundary labeled a , k , and $k-a$. The boundary is divided into three arcs labeled b , c , and b . A label $\frac{k}{2}$ is above the circle.

The rest of this subsection is devoted to the proof of the following:

Proposition 3.14. *If $p = 4r$, with $r \leq 7$ an odd prime, is generic then the null vector $v_0 \in V_{p,3}$ is cyclic in genus 3.*

We already know from Proposition 3.4 that $Z_{p,3}$ is included in the cyclic subspace generated by the null vector. When p is generic, its orthogonal is spanned by vectors u_σ with σ a coloration of the Tetrahedron graph of type *I* or *II* given in Proposition 3.10. We must show that these vectors also belong to the cyclic space generated by v_0 to conclude.

We split the proof into four lemmas which, together, imply Proposition 3.14.

Lemma 3.15. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type *I*, as defined in Proposition 3.10, such that $a \neq \frac{k}{2}$ and $b \neq \frac{k}{2}$. Then u_σ belongs to the cyclic space generated by v_0 .*

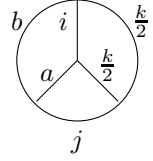
Proof. The proof relies on the following remark: embed a colored Tetrahedron graph in H_3 , choose two opposite edges of the graph and perform two Whitehead moves on these edges as in the fusion rule in Lemma 2.1. You get this way another embedding of the Tetrahedron graph inside H_3 . While choosing the edges colored by b and its opposite colored by $\frac{k}{2}$ in a type *I* coloration of

the Tetrahedron graph, we get:

$$\begin{array}{c} \frac{k}{2} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ a \quad b \quad c \end{array} = \sum_{i,j} \alpha_{i,j} \begin{array}{c} b \quad i \quad \frac{k}{2} \\ \diagdown \quad \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ a \quad \quad j \end{array}$$

where $\alpha_{i,j} = \left\langle \begin{array}{c} a \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ i \quad \frac{k}{2} \quad b \end{array} \right\rangle \left\langle \begin{array}{c} \frac{k}{2} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ j \quad c \quad a \end{array} \right\rangle c_{i,j}$ with $c_{i,j} \neq 0$.

When $i = \frac{k}{2}$, we have $\alpha_{i,j} = 0$ by Proposition 3.10. When $i \neq \frac{k}{2}$, the vector



belongs to $Z_{p,3}$ and thus to the cyclic space generated by v_0 by Proposition 3.4. This concludes the proof. \square

Lemma 3.16. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type I, as defined in Proposition 3.10, such that $a = b = \frac{k}{2}$. Then u_σ belongs to the cyclic space generated by v_0 .*

Proof. Using Lemma 2.1, we get:

$$v_a := \begin{array}{c} \frac{k}{2} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ a \quad \frac{k}{2} \quad \frac{k}{2} \end{array} = \left\langle \begin{array}{c} \frac{k}{2} \quad \frac{k}{2} \\ \text{---} \text{---} \\ \frac{k}{2} \end{array} \right\rangle \begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ a \end{array} + v'$$

, where v' is a vector orthogonal to the first one.

Now according to Proposition 3.4, we have that the null vector is cyclic in genus 2. This implies

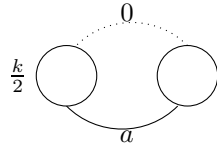
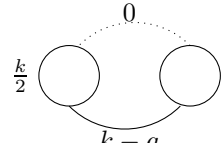
that the vector $\begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ a \end{array}$ belongs to the cyclic space generated by the null vector in

genus 3.

When $a = \frac{k}{2}$, then $W_{[v_a]}$ is one dimensional, so according to Lemma 3.5, either $v_{\frac{k}{2}}$ belongs the cyclic space generated by v_0 or it belongs to its orthogonal. But its scalar product with the vector

$\begin{array}{c} 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ a \end{array}$ is a non null $3j$ -symbol. Thus $v_a \in \mathcal{A}_{p,3} \cdot v_0$.

When $a \neq \frac{k}{2}$, then $W_{[v_a]}$ is two dimensional generated by v_a and v_{k-a} . If $v = \alpha_1 v_a + \alpha_2 v_{k-a}$ belongs to the orthogonal of the cyclic space generated by v_0 , then v is orthogonal to both vectors


 and
 
 . This implies that $v = 0$ so $W_{[v_a]}$ is included in the cyclic space of the null vector.

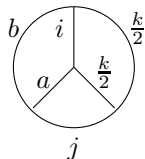
□

Lemma 3.17. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type II, as defined in Proposition 3.10, such that either we have $a \neq b$ and $a \neq k - b$, or we have $c \equiv \frac{k}{2} \pmod{4}$. Then u_σ belongs to the cyclic space generated by v_0 .*

Proof. The proof is similar to the proof of Lemma 3.15. Using Lemma 2.1 twice, we get:

$$\begin{array}{c} \frac{k}{2} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ b \quad \quad b \end{array} = \sum_{i,j} \alpha_{i,j} \begin{array}{c} b \quad j \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ a \quad \quad k-a \\ i \end{array}$$

where $\alpha_{i,j} = \left\langle \begin{array}{c} i \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ b \quad \quad b \\ a \quad \quad k-a \end{array} \right\rangle \left\langle \begin{array}{c} j \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ b \quad \quad \frac{k}{2} \\ k-a \end{array} \right\rangle c_{i,j}$ with $c_{i,j} \neq 0$.

When $i = \frac{k}{2}$, we have $\alpha_{i,j} = 0$ by Proposition 3.10. When $i \neq \frac{k}{2}$, using the fact that either $a \neq b$ and $a \neq k - b$, or $c \equiv \frac{k}{2} \pmod{4}$, we see that the vector  belongs to $Z_{p,3}$ and

thus to the cyclic space generated by v_0 by Proposition 3.4. This concludes the proof. □

Lemma 3.18. *Suppose $p = 4r$ with $r \geq 7$ an odd prime. Let σ be a coloration of the Tetrahedron graph of type II, as defined in Proposition 3.10, such that we have $a = b$ or $a = k - b$ and $c \equiv \frac{k}{2} + 2 \pmod{4}$. Then u_σ belongs to the cyclic space generated by v_0 .*

Proof. Using Lemma 2.1, we get:

$$\begin{array}{c} \frac{k}{2} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ a \quad \quad k-a \\ a \quad \quad c \end{array} = \sum_i \left\{ \begin{array}{cc} a & a \\ k-a & a \end{array} \begin{array}{c} \frac{k}{2} \\ i \end{array} \right\} a \begin{array}{c} i \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ a \quad \quad k-a \\ c \end{array}$$

Let $T \in \widetilde{\text{Mod}}(\Sigma_g)$ represent a lift of the Dehn twist around the edge colored by i in the above graph. We have:

$$\rho_{p,3}(T) \cdot \begin{array}{c} \frac{k}{2} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{k-a} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{c} \end{array} = \sum_i \left\{ \begin{array}{ccc} a & a & \frac{k}{2} \\ k-a & a & i \end{array} \right\} (-1)^i A^{i(i+2)} a \begin{array}{c} i \\ \diagup \quad \diagdown \\ \text{a} \quad \text{k-a} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{c} \end{array}$$

In particular, $\rho_{p,3}(T) \cdot u_\sigma$ belongs to the space generated by the vectors of the form

$$\begin{array}{c} j \\ \diagup \quad \diagdown \\ \text{a} \quad \text{k-a} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{c} \end{array}.$$

Whenever $j \neq \frac{k}{2}$, these generating vectors belong to $Z_{p,3}$ and thus to the cyclic space generated by the null vector. Denote by β the scalar product $\langle u_\sigma, \rho_{p,3}(T) u_\sigma \rangle$.

If $\beta = 0$, then $\rho_{p,3}(T) \cdot u_\sigma$ belongs to the cyclic space generated by v_0 , so does u_σ for $\rho_{p,3}(T)$ is invertible.

If $\beta \neq 0$, then the operator $a := \beta \cdot \mathbf{1} + \left\{ \begin{array}{ccc} a & a & \frac{k}{2} \\ k-a & a & \frac{k}{2} \end{array} \right\} \rho_{p,3}(T) \in \mathcal{A}_{p,3}$ is invertible since $\rho_{p,3}(T)$ has finite order. Since $a \cdot u_\sigma$ belongs to the cyclic space generated by the null vector, so does u_σ . \square

4 Decomposition into irreducible factors

In this section, we will prove Theorems 1.1 and 1.2. Denote by $(\mathcal{A}_{p,g})'$ the commutant of the algebra $\mathcal{A}_{p,g}$, i.e. the subspace of $\text{End}(V_{p,g})$ of operators commuting with all the $\rho_{p,g}(\phi)$ for $\phi \in \widetilde{\text{Mod}}(\Sigma_g)$.

The dimension of $(\mathcal{A}_{p,g})'$ is equal to the number of irreducible submodules of $V_{p,g}$. We thus have to show that $\dim((\mathcal{A}_{p,g})')$ is one if $p = 2r^2$ and $p = 2r_1 r_2$ and is two when $p = 4r$ with the additional assumptions of the two theorems.

Consider the following linear map:

$$f : \left\{ \begin{array}{ccc} (\mathcal{A}_{p,g})' & \hookrightarrow & V_{p,g} \\ \theta & \longmapsto & \theta \cdot v_0 \end{array} \right.$$

The cyclicity of v_0 (Proposition 3.4) implies that f is injective. Moreover if $\phi \in \widetilde{\text{Mod}}(\Sigma_g)$ is the lift of a homeomorphism of Σ_g that extends to H_g through $\alpha : \Sigma_g \rightarrow \partial H_g$, then:

$$\rho_{p,g}(\phi) \cdot v_0 = v_0$$

Denote by $\widetilde{\text{Mod}}(H_g) \subset \widetilde{\text{Mod}}(\Sigma_g)$ the subgroup generated by these ϕ . By definition, we have:

$$\text{Range}(f) \subset \left\{ v \in V_{p,g} \text{ so that } \rho_{p,g}(\phi) \cdot v = v, \text{ for all } \phi \in \widetilde{\text{Mod}}(H_g) \right\}$$

In particular, for any trivalent graph Γ , we have $\text{Range}(f) \subset W_{[0]}(\Gamma)$ where $[0]$ is the class of the coloring sending every edges of Γ to 0. As an immediate consequence, we get the:

Proof of Theorems 1.1 and 1.2 when $p = 2r^2$. When $p = 2r^2$, with r an odd prime, then $W_{[0]}$ is one-dimensional, generated by v_0 . Thus $\text{Range}(f) = \{v_0\}$ and $(\mathcal{A}_{2r^2,g})' = \{\mathbf{1}\}$. The Schur lemma implies that the module $V_{2r^2,g}$ is irreducible. \square

Remark. When $p = 18$, we remark that the numbers $\mu_0, \mu_1, \dots, \mu_{23}$ are pairwise distinct. The proof of Roberts [Rob01] applies word-by-word in this case to show that $V_{18,g}$ is irreducible. Indeed the fact that the μ_i are distinct implies that the null vector $v_0 \in V_{18,1}$ is cyclic for the action of the group generated by the Dehn twist along the longitude of H_1 . This easily implies that $v_0 \in V_{18,g}$ is cyclic for the action of $\text{Mod}(\widehat{\Sigma_g})$ for arbitrary $g \geq 1$ and we conclude as above by noticing that $W_{[0]}$ is one dimensional generated by v_0 .

4.1 The case where $p = 4r$

Let $p \geq 3$ be such that $p \equiv 4 \pmod{8}$. Consider a link $L \subset \Sigma_g \times \{\frac{1}{2}\}$ inside the cylinder $\Sigma_g \times [0, 1]$ and color L by p parallel copies of ω or, equivalently, by the $(\frac{p-4}{2})$ -th Jones-Wenzl idempotent. The gluing of the above cobordism on H_g induces an operator acting on $V_{p,g}$. In [BHMV95] it is shown that this operator only depends the homology class of L in $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ and we get this way an injective morphism of algebras:

$$i : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \hookrightarrow \mathcal{A}_{p,g}$$

Its action on v_0 gives the space $W_{[0]} \cong \mathbb{C}[H_1(H_g, \mathbb{Z}/2\mathbb{Z})]$.

We denote by P the projector of $V_{p,g}$ on the subspace of vectors fixed by the operators of $i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$. Clearly $P \in (\mathcal{A}_{p,g})'$.

Note $x_i, y_i \in H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ the meridian and longitude around the i -th hole and note:

$$\Theta_i := \frac{1}{\sqrt{2}}(-1 + x_i + y_i + x_i y_i) \in \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})]$$

The Θ_i 's are symmetries which pairwise commute and

$$P = \frac{1}{2^g} \left(\sqrt{2}(\Theta_1 + \dots + \Theta_g) + g + 1 \right)$$

The symmetric group σ_g acts by permutation on the generators of $\mathbb{C}[\Theta_1, \dots, \Theta_g]$. We note $\mathcal{W}_g \subset i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$ the subalgebra of $\mathbb{C}[\Theta_1, \dots, \Theta_g]$ of elements fixed by σ_g .

Finally we denote by $I \subset i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$ the ideal generated by the elements $(x_i - 1)$ for $1 \leq i \leq g$. We have:

$$\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] / I \cong \mathbb{C}[H_1(H_g, \mathbb{Z}/2\mathbb{Z})] \cong W_{[0]}$$

Lemma 4.1. *Consider the action of $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ on $i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})])$. Then:*

1. *The vectors fixed by this action are the ones of $\text{Span}(\mathbb{1}, P)$.*
2. *For every $w \in \mathcal{W}_g$ and $\phi \in Sp(2g, \mathbb{Z}/2\mathbb{Z})$ we have:*

$$\phi \cdot w - w \in I$$

Proof. The first point follows from the fact that the action of $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ on $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ has two orbits: the singleton containing the neutral element and the set containing the other elements. Indeed by taking an appropriate $\mathbb{Z}/2\mathbb{Z}$ -basis of $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$, this action is described by the usual Birman generators of $Sp(2g, \mathbb{Z})$ ([Bir71]) passed to the quotient in $Sp(2g, \mathbb{Z}/2\mathbb{Z})$, that is the $2g \times 2g$ matrices:

$$\begin{pmatrix} A & 0_g \\ 0_g & A^* \end{pmatrix}, \begin{pmatrix} 1_g & B \\ 0_g & 1_g \end{pmatrix} \text{ and } \begin{pmatrix} 0_g & 1_g \\ 1_g & 0_g \end{pmatrix}$$

where $A \in \text{GL}(g, \mathbb{Z}/2\mathbb{Z})$ and B is symmetric. We just have to remark that the commutant of the algebra generated by these matrices consists of the scalar matrices to conclude.

To prove the second point, denote by $X_i, Y_i, Z_{i,j}$ for $1 \leq i, j \leq g$ the class in $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ of the Dehn twists of Figure 6 generating $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$. First note that the operators Θ_i are invariant under the action of the X_i and Y_i and that the element of the algebra \mathcal{W}_g are invariant under permutation of the handles. We are reduced to show that for $w \in \mathcal{W}_g$, we have $Z_{1,2} \cdot w - w \in I$.

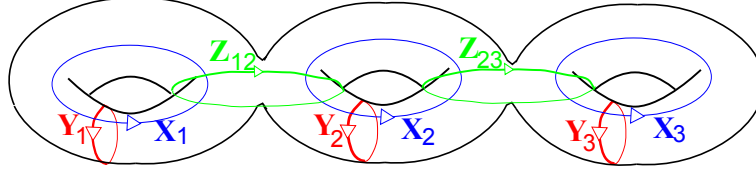


Figure 6: Some Dehn twists generating $Sp(2g, \mathbb{Z}/2\mathbb{Z})$ when $g = 3$ by passing to the quotient.

First note that $Z_{1,2} \cdot \Theta_i = \Theta_i$ when $i \notin \{1, 2\}$. Then we compute:

$$\begin{aligned}
Z_{1,2} \cdot \Theta_1 - \Theta_1 &= \frac{1}{\sqrt{2}}(y_1 + x_1 y_1)(x_2 - 1) \in I \\
Z_{1,2} \cdot \Theta_2 - \Theta_2 &= \frac{1}{\sqrt{2}}(y_2 + x_2 y_2)(x_1 - 1) \in I \\
Z_{1,2} \cdot (\Theta_1 \Theta_2) - (\Theta_1 \Theta_2) &= \frac{1}{2}((x_1 x_2 - 1)(x_1 y_1 + y_1)(x_2 y_2 + y_2) \\
&\quad + (x_2 - 1)(y_1 x_1 + y_1)(-1 + x_2) \\
&\quad + (x_1 - 1)(x_2 y_2 + y_2)(-1 + x_1)) \in I
\end{aligned}$$

□

The case $p = 4r$ of Theorems 1.1 and 1.2 are easily deduced from the:

Proposition 4.2. *If $p \equiv 4 \pmod{8}$ and $v_0 \in V_{p,g}$ is cyclic, then*

$$f^{-1}(\mathbb{C}[H_1(H_g, \mathbb{Z}/2\mathbb{Z})]) = \text{Span}(\mathbb{1}, P)$$

Proof. Let $\Theta \in (\mathcal{A}_{p,g})'$. Since $\Theta \cdot v_0$ lies in $W_{[0]}$ and is invariant under permutation of the handles, there exists an element $w \in \mathcal{W}_g$ such that $w \cdot v_0 = \Theta \cdot v_0$. Now if $\phi \in \widetilde{\text{Mod}}(\Sigma_g)$, then:

$$\Theta \circ \rho_{p,g}(\phi) \cdot v_0 = \rho_{p,g}(\phi) \circ \Theta \cdot v_0 = \rho_{p,g}(\phi) \circ w \cdot v_0 = w \circ \rho_{p,g}(\phi) \cdot v_0$$

where we used the second point of Lemma 4.1 in the last equality. Using the cyclicity of v_0 we get that $\Theta = w \in \mathcal{W}_g$. We conclude using the first point of Lemma 4.1. □

4.2 The case where $p = 2r_1 r_2$

In this subsection, we suppose that $p = 2r_1 r_2$ with r_1, r_2 distinct odd primes.

In this case, there exists a unique integer $x \in \{1, \dots, r_1 r_2 - 2\}$ such that $\mu_x = 1$. This integer is even and verifies either

$$\begin{cases} x \equiv -2 \pmod{r_1} \\ x \equiv 0 \pmod{r_2} \end{cases} \text{ or } \begin{cases} x \equiv 0 \pmod{r_1} \\ x \equiv -2 \pmod{r_2} \end{cases}.$$

We begin by stating a technical lemma which proof will be the subject of the next subsection:

Lemma 4.3. *If (x, x, x) is p -admissible, then we have the following:*

$$\begin{Bmatrix} x & x & 2 \\ x & x & 0 \end{Bmatrix} \begin{Bmatrix} x & x & 4 \\ x & x & x \end{Bmatrix} \neq \begin{Bmatrix} x & x & 4 \\ x & x & 0 \end{Bmatrix} \begin{Bmatrix} x & x & 2 \\ x & x & x \end{Bmatrix}$$

Lemma 4.4. *Let $p \geq 3$ be such that (x, x, x) is p -admissible. Let Γ_1, Γ_2 be two trivalent graphs which only differ by a single Whitehead move inside a ball B^3 as drawn in Figure 3. Then:*

$$W_{[0]}(\Gamma_1) \cap W_{[0]}(\Gamma_2) \subset \text{Span}(u_\sigma^{\Gamma_1}, \text{ such that } \sigma(a)\sigma(b)\sigma(c)\sigma(d) = 0)$$

Proof. Let σ_1, σ_2 be two p -admissible colorings of Γ_1 , with colors 0 or x , such that:

$$\sigma_1(e) = \sigma_2(e), \forall e \in E(\Gamma_1) - \{i\}$$

and with $\sigma_i(a) = \sigma_i(b) = \sigma_i(c) = \sigma_i(d) = x$ and $\sigma_1(i) = 0, \sigma_2(i) = x$.

Suppose there exists $(\alpha, \beta) \in \mathbb{C}^2$ so that:

$$v := \alpha u_{\sigma_1} + \beta u_{\sigma_2} \in W_{[0]}(\Gamma_2)$$

We must show that $\alpha = \beta = 0$ to conclude. Using Lemma (2.1), we get:

$$v = \left(\alpha \begin{Bmatrix} x & x & 2 \\ x & x & 0 \end{Bmatrix} + \beta \begin{Bmatrix} x & x & 2 \\ x & x & x \end{Bmatrix} \right) \begin{array}{c} \diagup \diagdown \\ 2 \end{array} + \left(\alpha \begin{Bmatrix} x & x & 4 \\ x & x & 0 \end{Bmatrix} + \beta \begin{Bmatrix} x & x & 4 \\ x & x & x \end{Bmatrix} \right) \begin{array}{c} \diagup \diagdown \\ 4 \end{array} + v'$$

where $\begin{array}{c} \diagup \diagdown \\ 2 \end{array}$ and $\begin{array}{c} \diagup \diagdown \\ 4 \end{array}$ represent the vectors associated to colorations of Γ_2 by the same colors that σ_1, σ_2 outside the ball B^3 and with the edge j colored respectively by 2 and 4.

The vector v' is orthogonal to the two previous ones.

Now since $v \in W_{[0]}(\Gamma_2)$, we have the following system:

$$\left(\begin{Bmatrix} x & x & 2 \\ x & x & 0 \\ x & x & 4 \\ x & x & 0 \end{Bmatrix} \quad \begin{Bmatrix} x & x & 2 \\ x & x & x \\ x & x & 4 \\ x & x & x \end{Bmatrix} \right) \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We conclude using Lemma 4.3 □

If $i \in \{1, \dots, g\}$, we note $b_i \in V_{p,g}$ the vector representing a single ribbon colored by x around the i -th hole.

Lemma 4.5. *If \mathcal{G}^g represents the set of all trivalent graph of genus g , then:*

$$\bigcap_{\Gamma \in \mathcal{G}^g} W_{[0]}(\Gamma) = \text{Span}(v_0, b_i, 1 \leq i \leq g)$$

Proof. Let σ be a coloring of the graph of Figure 7 such that:

1. $\sigma(e) \in \{0, x\}$, for all $e \in E(\Gamma)$,
2. There exists $i < j$ with $\sigma(a_i) = \sigma(b_i) = \sigma(a_j) = \sigma(b_j) = x$.

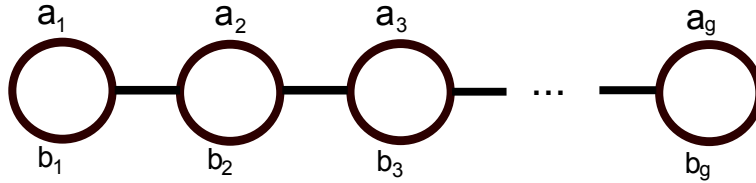


Figure 7: A trivalent graph of genus g .

We can suppose that for every $i < k < j$, then $\sigma(a_k)\sigma(b_k) = 0$. Using Lemma 4.4 with $a = a_i, b = a_j, c = b_i$ and $d = b_j$, we have that the projection of u_σ on $\bigcap_{\Gamma \in \mathcal{G}^g} W_{[0]}(\Gamma)$ is null.

We conclude by noticing that if σ is a coloring of Γ , with colors in $\{0, x\}$, that does not satisfies 2, then $u_\sigma = b_i$ for some $i \in \{1, \dots, g\}$ or $u_\sigma = v_0$. □

Lemma 4.6. *There exists an element $a \in \mathcal{A}_{2r_1r_2,1}$ so that:*

$$\begin{cases} a \cdot u_0 = u_x \\ a \cdot u_x = u_0 \end{cases}$$

Proof. It is enough to show that there exists a symmetry $\psi \in (\mathcal{A}_{2r_1r_2,1})'$ so that:

$$\psi \cdot u_0 = u_x \text{ and } \psi \cdot u_x = u_0$$

Indeed, the cyclicity of u_0 (Proposition 3.3) implies the existence of $a \in \mathcal{A}_{2r_1r_2,1}$ so that

$$a \cdot u_0 = u_x$$

If such a ψ does exist, we then have:

$$a \cdot u_x = a \circ \psi \cdot u_0 = \psi \circ a \cdot u_0 = u_0$$

The symmetry ψ is defined as follows: choose $i \in \{0, \dots, r_1r_2 - 2\}$, then only one of the following two cases occurs:

- Either there exists $j \in \{0, \dots, r_1r_2 - 2\}$ so that

$$\begin{cases} j \equiv i & (\text{mod } 2r_1) \\ j \equiv -i - 2 & (\text{mod } r_2) \end{cases}$$

and we put $\psi(u_i) := +u_j$.

- Or there exists $j \in \{0, \dots, r_1r_2 - 2\}$ so that

$$\begin{cases} j \equiv i & (\text{mod } 2r_2) \\ j \equiv -i - 2 & (\text{mod } r_1) \end{cases}$$

and we put $\psi(u_i) := -u_j$.

A straightforward computation shows that ψ commutes with $\rho_{p,1}(T)$ and $\rho_{p,1}(S)$ and either ψ or $-\psi$ sends u_0 to u_x . \square

The proof of Theorems 1.1 and 1.2 when $p = 2r_1r_2$ follows from the following:

Proposition 4.7. *Let r_1, r_2 be two distinct odd primes, $p = 2r_1r_2$ and $g \geq 2$ be such that $v_0 \in V_{p,g}$ is cyclic. Then $V_{p,g}$ is irreducible.*

Proof. Using Lemma 4.5 and the fact that the vectors of $\text{Range}(f)$ must be invariant under permutation of the handles, we have that:

$$\text{Range}(f) \subset \text{Span}(v_0, b_1 + \dots + b_g)$$

By contradiction, suppose there exists $\Theta \in (\mathcal{A}_{p,g})'$ so that:

$$\begin{aligned} \Theta \cdot v_0 &= b_1 + \dots + b_g \\ &= (a \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} + \dots + \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a) \cdot v_0 \end{aligned}$$

where $a \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}$ denotes the embedding of the element $a \in \mathcal{A}_{p,1}$, seen as a linear combination of ω -colored link in $\Sigma_1 \times [0, 1]$, in the first handle of $\Sigma_g \times [0, 1]$.

Now we have:

$$\begin{aligned} \Theta^2 \cdot v_0 &= (a \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} + \dots + \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a)^2 \cdot v_0 \\ &= gv_0 + (a \otimes a \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}) \cdot v_0 + \dots + (\mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a \otimes a) \cdot v_0 \end{aligned}$$

We see that $\Theta^2 \cdot v_0$ does not belong to $\bigcap_{\Gamma \in \mathcal{G}^g} W_{[0]}(\Gamma)$ which contradicts the fact that $\Theta^2 \in (\mathcal{A}_{p,g})'$. \square

4.3 Proof of Lemma 4.3

In this subsection we put $p = 2r_1r_2$, with r_1, r_2 two distinct odd primes. We suppose there exists $x \in \{1, \dots, r_1r_2 - 2\}$ so that (x, x, x) is p -admissible and so that

$$\begin{cases} x \equiv 0 & (\text{mod } 2r_1) \\ x \equiv -2 & (\text{mod } r_2) \end{cases}$$

We also choose A_1 and A_2 some primitive r_1 -th and r_2 -th roots of unity, so that $A^2 = A_1A_2$. In particular we have $A^{2x} = A_2^{-2}$.

The goal of this subsection is to show that:

$$D := \begin{Bmatrix} x & x & 2 \\ x & x & 0 \end{Bmatrix} \begin{Bmatrix} x & x & 4 \\ x & x & x \end{Bmatrix} - \begin{Bmatrix} x & x & 4 \\ x & x & 0 \end{Bmatrix} \begin{Bmatrix} x & x & 2 \\ x & x & x \end{Bmatrix} \neq 0$$

A straightforward computation, using the formula of the recoupling coefficients ([MV94]), gives:

$$\begin{aligned} D &= (-1)^{\frac{x}{2}+1} \frac{[3][5]![x] \left[\frac{3}{2}x+1\right]! \left(\left[\frac{x}{2}\right]!\right)^3}{[2][x+3]!([x+2]!)^2[x+3] \left[\frac{x}{2}+1\right]} \left(\left[\frac{x}{2}-1\right]^2 \left[\frac{x}{2}\right]^2 \left[\frac{x}{2}+1\right] \right. \\ &\quad - [2]^2 \left[\frac{x}{2}\right]^3 \left[\frac{3}{2}+2\right] \left[\frac{x}{2}+1\right] + \left[\frac{x}{2}-1\right] \left[\frac{x}{2}\right] \left[\frac{x}{2}+1\right] \left[\frac{x}{2}+2\right] \left[\frac{x}{2}+3\right] \\ &\quad \left. + [x-1][x+3] \left[\frac{x}{2}\right]^2 \left[\frac{x}{2}+1\right] - [x-1] \left[\frac{3}{2}x+2\right] [x+3] \right) \\ &= (-1)^{\frac{x}{2}+1} \frac{[3][5]![x] \left[\frac{3}{2}x+1\right]! \left(\left[\frac{x}{2}\right]!\right)^3}{[2][x+3]!([x+2]!)^2[x+3] \left[\frac{x}{2}+1\right] (A^2 - A^{-2})^7 A_1^{10} A_2^9} \cdot P(A_1, A_2) \end{aligned}$$

where we put:

$$\begin{aligned} P(x, y) &:= x^{20}y^{16} - x^{17}y^{17} - x^{16}y^{18} + x^{19}y^{13} - 4x^{18}y^{14} + 3x^{17}y^{15} + 2x^{15}y^{17} - x^{19}y^{11} \\ &\quad - 5x^{17}y^{13} - 4x^{15}y^{15} + 4x^{14}y^{16} - 2x^{13}y^{17} - x^{18}y^{10} + 2x^{17}y^{11} + 6x^{16}y^{12} + 2x^{15}y^{13} \\ &\quad - x^{14}y^{14} + 2x^{13}y^{15} + x^{11}y^{17} + 2x^{15}y^{11} + x^{14}y^{12} + x^{13}y^{13} - 6x^{12}y^{14} + x^{10}y^{16} \\ &\quad + x^{17}y^7 + 4x^{16}y^8 - x^{15}y^9 - 4x^{14}y^{10} + 4x^{12}y^{12} + x^{15}y^7 + x^{13}y^9 - 4x^{12}y^{10} + x^{11}y^{11} \\ &\quad + 4x^{10}y^{12} - 4x^8y^{14} - x^7y^{15} - 2x^{15}y^5 - 6x^{14}y^6 - 4x^{13}y^7 + 2x^{12}y^8 - 8x^{11}y^9 - 6x^{10}y^{10} \\ &\quad - 6x^9y^{11} - x^7y^{13} + x^{13}y^5 + 6x^{11}y^7 + 6x^{10}y^8 + 8x^9y^9 - 2x^8y^{10} + 4x^7y^{11} + 6x^6y^{12} + 2x^5y^{13} \\ &\quad + x^{13}y^3 + 4x^{12}y^4 - 4x^{10}y^6 - x^9y^7 + 4x^8y^8 - x^7y^9 - x^5y^{11} - 4x^8y^6 + 4x^6y^8 + x^5y^9 - 4x^4y^{10} - x^3y^{11} \\ &\quad - x^{10}y^2 + 6x^8y^4 - x^7y^5 - x^6y^6 - 2x^5y^7 - x^9y - 2x^7y^3 + x^6y^4 - 2x^5y^5 - 6x^4y^6 - 2x^3y^7 + x^2y^8 + 2x^7y \\ &\quad - 4x^6y^2 + 4x^5y^3 + 5x^3y^5 + xy^7 - 2x^5y - 3x^3y^3 + 4x^2y^4 - xy^5 + x^4 + x^3y - y^2 \end{aligned}$$

Note that $P(x, y)$ does not depend on r_1, r_2 or x . The proof reduces to show that $P(A_1, A_2) \neq 0$ for A_1, A_2 any primitive r_1 -th and r_2 -th roots of unity.

Consider the following algebraic curves in \mathbb{C}^2 :

$$\begin{aligned} \mathcal{C} &:= \{(z_1, z_2) \in \mathbb{C}^2 \text{ so that } P(z_1, z_2) = 0\} \\ \mathcal{T} &:= \{(z_1, z_2) \in \mathbb{C}^2 \text{ so that } |z_1|^2 = |z_2|^2 = 1\} \end{aligned}$$

Note that \mathcal{T} is a torus, has degree 3 and that these two curves share no irreducible components in common. The Bézout theorem (see [Har77] Chap I Corollary 7.8) implies that:

$$\#(\mathcal{C} \cap \mathcal{T}) \leq \deg(\mathcal{C}) \cdot \deg(\mathcal{T}) = 108$$

Now suppose there exist A_1 and A_2 some primitive r_1 and r_2 roots of unity so that $P(A_1, A_2) = 0$. Since $P(x, y) \in \mathbb{Z}[x, y]$, the equality $P(A_1, A_2) = 0$ must hold for every r_1 and r_2 roots of unity. Thus we have:

$$r_1 \cdot r_2 \leq \#(\mathcal{C} \cap \mathcal{T}) \leq \deg(\mathcal{C}) \cdot \deg(\mathcal{T}) = 108$$

So we just have the following possible cases:

$$\{r_1, r_2\} \in \{\{3, 5\}, \{5, 7\}, \{3, 11\}, \{5, 11\}, \{7, 11\}, \{3, 13\}, \{5, 13\}, \{7, 13\}\}$$

First if $\{r_1, r_2\} = \{3, 5\}, \{5, 7\}, \{3, 11\}$ or $\{5, 13\}$, then $x = 10, 28, 22$ and 50 respectively and we see that (x, x, x) is not $2r_1 r_2$ -admissible.

We handle the four reminding cases by checking that $P(e^{\frac{2i\pi}{5}}, e^{\frac{2i\pi}{11}}) \neq 0$, $P(e^{\frac{2i\pi}{7}}, e^{\frac{2i\pi}{11}}) \neq 0$, $P(e^{\frac{2i\pi}{3}}, e^{\frac{2i\pi}{13}}) \neq 0$ and $P(e^{\frac{2i\pi}{7}}, e^{\frac{2i\pi}{13}}) \neq 0$.

This concludes the proof of Lemma 4.3.

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